Week 4 Worksheet
Free Electron Gas

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Exercise 1. Suppose you have N electrons in a box of side length L.

a) Show that the Fermi energy is

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3N\pi^2}{L^3}\right)^{2/3}.$$

b) Find the total energy of the electrons in terms of E_F.

Exercise 2. Now, consider a free electron gas in two dimensions, confined to a square of side length L.

a) Find the Fermi energy in terms of N and L, and show that the average energy of the particles is $E_F/2$.

b) Let $g(E)\,\mathrm{d}E$ be the number of particles with energy E in the interval $\mathrm{d}E$. $g(E)$ is called the **density of states** and is useful in various problems in quantum statistical mechanics. Calculate $g(E)$ for the particles. Your formula should be constant, i.e. independent of E.

Exercise 3. A white dwarf star is basically a free electron gas, with a bunch of nuclei mixed in to balance the charge and to provide the gravitational attraction that holds it all together.

a) Use dimensional analysis to argue that the gravitational potential energy of a uniform-density sphere (mass M, radius R) must be

$$E_{\text{grav}} = -\alpha\frac{GM^2}{R},$$

where $\alpha > 0$ is some numerical constant. Be sure to explain the minus sign. It turns out that $\alpha = 3/5$, which you can derive by calculating the work needed to assemble the sphere shell-by-shell.

b) Assuming that the star contains one proton and one neutron for each electron and that the electrons are nonrelativistic, find the total electron kinetic energy (use Exercise 1 to make your life easier).

c) The equilibrium radius of the star is that which minimizes the total energy $E_{\text{grav}} + E_{\text{k}}$. Sketch the total energy as a function of R, and find a formula for the equilibrium radius in terms of the mass. As the mass increases, does the radius increase or decrease? Does this make sense?

d) If $M = 2 \cdot 10^{30}$ kg, the mass of the sun, evaluate the equilibrium radius and the density. Compare the density to that of water.